

Extreme throat initial data set and horizon area–angular momentum inequality for axisymmetric black holes

Sergio Dain

*Facultad de Matemática, Astronomía y Física, FaMAF, Universidad Nacional de Córdoba,
Instituto de Física Enrique Gaviola, IFEG, CONICET,
Ciudad Universitaria (5000) Córdoba, Argentina and*

Max Planck Institute for Gravitational Physics (Albert Einstein Institute) Am Mühlenberg 1 D-14476 Potsdam Germany.

(Dated: November 4, 2010)

We present a formula that relates the variations of the area of extreme throat initial data with the variation of an appropriate defined mass functional. From this expression we deduce that the first variation, with fixed angular momentum, of the area is zero and the second variation is positive definite evaluated at the extreme Kerr throat initial data. This indicates that the area of the extreme Kerr throat initial data is a minimum among this class of data. And hence the area of generic throat initial data is bounded from below by the angular momentum. Also, this result strongly suggests that the inequality between area and angular momentum holds for generic asymptotically flat axially symmetric black holes. As an application, we prove this inequality in the non trivial family of spinning Bowen-York initial data.

PACS numbers: 04.70.Bw, 04.20.Ex, 02.30.Jr, 02.30.Xx, 02.40.Vh

I. INTRODUCTION

Geometrical inequalities play an important role in General Relativity, in particular for vacuum black holes, where the geometrical aspects of the theory appears in their pure form. A geometrical inequality in General Relativity relates quantities that have both a physical interpretation and a geometrical definition. The most relevant example is the positive mass theorem. The mass of the spacetime measures the total amount of energy and hence it should be positive from the physical point of view. Also, the mass m in General Relativity is represented by a pure geometrical quantity on a Riemannian manifold [1] [5]. From the geometrical mass definition, without the physical picture, it would be very hard to conjecture that this quantity should be positive. On the other hand, the highly non trivial proof of this inequality [34][35][39] reveals the subtle way in which Einstein equations describe the gravitational field.

For black holes, the first example of geometrical inequality is the Penrose inequality (see the recent review article [32] and references therein) which relates the area of the horizon A (the ‘size’ of the black holes) with the total mass of the spacetime

$$\sqrt{\frac{A}{16\pi}} \leq m. \quad (1)$$

Another example is the inequality between mass and the angular momentum J for axially symmetric black holes

$$\sqrt{|J|} \leq m. \quad (2)$$

See [22][17] [13], and also the generalization which includes charge presented in [14] [15]. Inequalities (1) and (2) are closed related with the weak cosmic censorship conjecture. They can be interpreted as indirect but relevant indications of the validity of this conjecture. As

in the case of the positive of the mass, these inequalities has been discovered first by physical arguments, and then proved afterward (under appropriated and restricted assumptions, see the references mentioned above) as a rigorous consequence of Einstein equations. The proofs provide also new insight into the mechanisms of Einstein equations. As an example (which is connected with the subject of this article), we mention that the proof of (2) involves a variational characterization of the extreme Kerr black hole as an absolute minimum of the mass.

The total mass is a global quantity. On the other hand the area A and the angular momentum in axial symmetry J , involved in inequalities (1) and (2) respectively, are quasi-local quantities. Namely they carry information on a bounded region of the spacetime. In contrast with a local quantity like a tensor field which depends on a point of the spacetime. Inequalities (1) and (2) relate global quantities with quasi-local ones. It is well known that the energy of the gravitational field can not be represented by a local quantity. The best one can hope is to obtain an expression of the total energy of a bounded region of the spacetime. These are the so called quasi-local mass definition (see the review article [37] and reference therein). For some of the quasi-local mass proposals there exist also positivity proofs and hence we obtain a quasi-local geometrical inequality. One would expect that for a black hole pure quasi-local inequalities are also valid. The relevance of this kind of inequalities is that they provide a much finer control on the dynamics of a black holes than the global versions. The main purpose of this article is to study one example of such quasi-local inequality for vacuum black holes and to give non trivial evidences of its validity.

The area of the horizon is a well defined quasi-local quantity for a generic black hole. In general, the quasi-local angular momentum is difficult to define (see [37]), but in the case of axial symmetry there exists a well

defined notion, namely the Komar angular momentum. This is the angular momentum used in inequality (2). That is, for generic (i.e. not necessarily stationary) axially symmetric black holes we have two well defined quasi-local physical quantities, the horizon area A and the angular momentum J . In terms of A and J , the Christodoulou [11] mass of the black hole is defined as follows

$$m_{bh} = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}. \quad (3)$$

For the Kerr black hole this formula gives precisely the total mass of the black hole which is equal to the total mass of the spacetime. In general, for an horizon with only one connected component one would expect that m_{bh} is less than the total mass of the spacetime with equality only for the Kerr black hole. This would give a generalization of Penrose inequality including angular momentum (in fact, an strong generalization) in the spirit of [31] (see the discussion in [32]). For the case of many black holes the negative interaction energy between the black holes should be also considered and hence it is not expected a simple inequality with respect to the total mass (see [38] for a discussion of the analog of this inequality with charges). We will not discuss this point further, since we are interested in this article only in quasi-local inequalities. We mention it because it plays a role in the physical interpretation of the quasi-local mass m_{bh} .

The formula (3) trivially satisfies the inequality

$$\sqrt{|J|} \leq m_{bh}. \quad (4)$$

This is, of course, just because the Kerr black hole satisfies this bound. Hence, if we accept (3) as the correct formula for the quasi-local mass of an axially symmetric black hole, then (4) provides the, rather trivial, quasi-local version of (2). The real question is, does the formula (3) represents the quasi-local mass of a non-stationary black hole? Let us analyze the behavior of the mass (3) from a physical point of view.

Let us assume that for a generic axially symmetric black hole the quantity m_{bh} gives a measure of the quasi-local mass of the black hole. Consider the evolution of m_{bh} . By the area theorem, we know that the horizon area will increase. If we assume axial symmetry, then the angular momentum will be conserved at the quasi-local level (we are assuming pure vacuum). On physical grounds, one would expect that in this situation the quasi-local mass of the black hole should increase with the area, since there is no mechanism at the classical level to extract mass from the black hole. In effect, the only way to extract mass from a black hole is by extracting angular momentum through a Penrose process. But angular momentum transfer is forbidden in pure vacuum axial symmetry. Then, one would expect that both the area A and the quasi-local mass m_{bh} should monotonically increase with time.

Let us take a time derivative of m_{bh} (denoted by a dot). From the formula (3) we obtain

$$\dot{m}_{bh} = \frac{\dot{A}}{32\pi m_{bh}} \left(1 - \left(\frac{8\pi J}{A} \right)^2 \right), \quad (5)$$

where we have used that the angular momentum J is conserved. Since, by the area theorem, we have

$$\dot{A} \geq 0, \quad (6)$$

the time derivative of m_{bh} will be positive (and hence the mass m_{bh} will increase with the area) if and only if the following inequality is satisfied

$$8\pi|J| \leq A. \quad (7)$$

Then, it is natural to conjecture that (7) should be satisfied for any horizon in an axially symmetric asymptotically flat initial data. If there are initial data that violate (7) then in the evolution the area will increase but the mass m_{bh} will decrease. This will indicate that the quantity m_{bh} has not the desired physical meaning. Also, a rigidity statement is expected. Namely, the equality in (7) is reached only by the extreme Kerr black hole.

If inequality (7) is true, then we have a non trivial monotonic quantity (in addition to the black hole area) m_{bh}

$$\dot{m}_{bh} \geq 0. \quad (8)$$

It is important to emphasize that the physical arguments presented above in support of (7) are certainly weaker in comparison with the ones behind the inequalities (1) and (2). A counter example of any of these inequality will prove that the standard picture of the gravitational collapse is wrong. On the other hand, a counter example of (7) will just prove that the quasi-local mass (3) is not appropriate to describe the evolution of a non-stationary black hole. One can imagine other expressions for quasi-local mass, may be more involved, in axial symmetry. On the contrary, reversing the argument, a proof of (7) will certainly prove that the mass (3) has physical meaning for non-stationary black holes as a natural quasi-local mass. Also, the inequality (7) provide a non trivial control of the size of a black hole valid at any time.

If the rigidity statement also holds, this inequality will provide a remarkable quasi-local measure of how far is the data from the extreme black hole data. This provides an ‘extremality criteria’ in the spirit of [8], although restricted only to axial symmetry. In the article [21] it has been conjectured that, within axial symmetry, to prove the stability of a nearly extreme black hole is perhaps simpler than a Schwarzschild black hole. It is possible that this quasi-local extremality criteria will have relevant applications in this context.

Let us also point out that the inequality (7) is related with the surface gravity density (or temperature) of a black hole. The surface gravity density κ of the Kerr

black hole can be written in terms of the quasi-local quantities A and J as follows

$$\kappa = \frac{1}{4\sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}}} \left(1 - \left(\frac{8\pi J}{A} \right)^2 \right). \quad (9)$$

From this equation, we see that for the Kerr black hole κ is positive because inequality (7) holds. Hence, if inequality (7) holds for generic, non-stationary axially symmetric black holes we can define the same expression for κ for this class of black holes.

All the previous arguments lead to the following conjecture

Conjecture I.1. *Consider an asymptotically flat, vacuum, complete axially symmetric initial data set for the Einstein equations. Then the following inequality holds*

$$8\pi|J| \leq A, \quad (10)$$

where A and J are the area and angular momentum of a connected component of the apparent horizon.

Note that in the previous discussion we have considered the area A of the event horizon (since we have used the area theorem). As it usual in geometrical inequalities in order to make an useful statement we need to replace the event horizon with a quasi-local quantity. In our case the most appropriate quantity appears to be the apparent horizon on the initial data. Generalization of the area theorem also holds (under appropriate assumptions) for apparent horizons (see the review article [3] and reference therein).

Let us mention some support for this inequality. This inequality has been proved for stationary black holes surrounded by matter in [29] [30] (also with charge). Although this case is only slightly related with the conjecture (since the conjecture applies for non-stationary spacetimes in vacuum) it is highly non trivial and it certainly suggests the validity the conjecture. It is also important to note that there is a counter example for this inequality in the non-asymptotically flat case [8]. That is, the assumption that the horizon belong to an asymptotically flat data is essential. This counter example points out that although the inequality involves only quasi-local quantities is not pure quasi-local in the sense that a global assumption should be made on the initial data (namely, asymptotic flatness).

The purpose of this article is to present non trivial evidences for the validity of conjecture I.1. The main part of this evidence is a formula that relates in a remarkable way the variations of the area and the variations of an appropriate defined mass functional on extreme throat initial data. These kind of data (described in section II) isolate the cylindrical structure of extreme black hole and hence they represent a natural source of counter examples to the inequality (10) as we will see. The very existence of this formula suggests that the inequality (10) should hold, at least in a relevant family of initial conditions.

Using this result we also prove the inequality in the non trivial family of spinning Bowen-York initial data.

The plan of the article is the following. In section II we describe extreme throat initial data. In section III we present our main result, given by theorem III.1. The proof of this result is divided naturally in two steps, described in section IV and V. In section IV we present an appropriate mass functional for extreme throat initial data. We calculate the first and second variations of this functional evaluated at the extreme Kerr cylindrical initial data. These results are analogous to the ones described for asymptotically flat axially symmetric initial data described in [19] [18]. Section V constitutes the most important part of the article. In this section it is shown the relation between the variations of the mass functional and the variations of the area. This relation was, in our opinion, completely unexpected a priori. In section VI we apply this result to prove the inequality (10) on the spinning Bowen-York black hole family. We discuss the relevant open problems in section VII. Finally, we conclude with an appendix in which we collect some properties of the extreme Kerr throat initial data.

II. EXTREME THROAT INITIAL DATA SET

In order to present our results we need to discuss first extreme throat initial data. The definition of this kind of initial data is motivated by the behavior of the Kerr black hole initial data in the extreme limit. Let us briefly review this behavior (for more details, see for example, section 2 in [23]).

Consider the Kerr black hole with mass m and angular momentum J . We define the following parameter μ (which has unit of mass) in terms of m and J

$$\mu = \sqrt{m^2 - |J|}. \quad (11)$$

The extreme Kerr black hole corresponds to $\mu = 0$. For the Schwarzschild black hole we have $\mu = m$.

In the standard Boyer-Lindquist coordinates for the Kerr black hole, take a slice $t = \text{constant}$. Let us denote by S the 3-dimensional manifold defined by that slice. The topology of this surface is $S = S^2 \times \mathbb{R}$. The triple (S, h_{ij}, K_{ij}) , where h_{ij} is the induced intrinsic metric on S and K_{ij} is the second fundamental form of S , constitute an initial data set for Einstein equations. That is, they are solutions of the constraint equations

$$D_j K^{ij} - D^i K = 0, \quad (12)$$

$$R - K_{ij} K^{ij} + K^2 = 0, \quad (13)$$

where D and R are the Levi-Civita connection and the Ricci scalar associated with h_{ij} , and $K = K_{ij} h^{ij}$. In these equations the indices are moved with the metric h_{ij} and its inverse h^{ij} .

For $\mu > 0$ these data have the geometry of two asymptotically flat ends. In the extreme limit $\mu = 0$ the geometry changes. One of the asymptotic ends is asymptotically flat but the other is cylindrical. Let us take a closer

look at the structure of the cylindrical end. In coordinates (r, θ, ϕ) , the induced metric on S has the form

$$h_{ij} = \Phi^4 \tilde{h}_{ij}, \quad (14)$$

where the conformal metric \tilde{h}_{ij} is defined by

$$\tilde{h} = e^{2q}(dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\phi^2, \quad (15)$$

and the functions Φ and q are given by equations (A4) in appendix A. The extrinsic curvature is given by

$$K_{ij} = \frac{2}{\eta} S_{(i} \eta_{j)}, \quad S_i = \frac{1}{\eta} \epsilon_{ijk} \eta^j \partial^k \omega, \quad (16)$$

where η^i is the axial Killing vector

$$\eta^i = \frac{\partial}{\partial \phi}, \quad (17)$$

the square of its norm η is given by (A1), ϵ_{ijk} denotes the volume element with respect to the metric h_{ij} and ω is given by (A3). The advantage of this particular form of writing K_{ij} is that it is easy to check from (16) that K_{ij} satisfies the momentum constraint (12) (see, for example, the appendix in [16] and [23]). In particular, we have that K_{ij} is trace-free, namely

$$K = 0. \quad (18)$$

That is, these initial data are maximal surfaces.

In these coordinates, the asymptotically flat end of the metric (14) corresponds to the limit $r \rightarrow \infty$ and the cylindrical end corresponds to the limit $r \rightarrow 0$. The radial coordinate r is a good coordinate in the asymptotically flat end, since the metric and the extrinsic curvature are manifestly asymptotically flat with respect to these coordinates: they have the standard decay to the flat metric.

On the other hand, in the limit $r \rightarrow 0$ the conformal factor Φ blows up. This is, however, just a coordinate problem. To see this, define $s = -\ln r$, then the cylindrical end corresponds to $s \rightarrow \infty$, and the metric has the form

$$h^0 = (\sqrt{r}\Phi)^4 (e^{2q}(ds^2 + d\theta^2) + \sin^2 \theta d\phi^2). \quad (19)$$

The functions $\sqrt{r}\Phi$ and q are smooth and uniformly bounded in the whole range $-\infty < s < \infty$.

The metric (19) and the second fundamental form (16) have a well defined limit $s \rightarrow \infty$ as initial data. For the metric h^0 in the limit $s \rightarrow \infty$ we obtain

$$h = \varphi_0^4 (e^{2q_0}(ds^2 + d\theta^2) + \sin^2 \theta d\phi^2). \quad (20)$$

where φ_0 and q_0 are defined by the limits (A5)–(A6). The extrinsic curvature K^{ij} has the form (16) where ω is replaced by its limiting value ω_0 defined by (A7) and all the other quantities are computed with respect to the metric (20). These are in fact solutions of the constraint equations (12)–(13) on $S^2 \times \mathbb{R}$. We call these initial data set the extreme Kerr throat initial data set.

Let us make a summary. The extreme Kerr throat initial data set is constructed out of the Kerr black hole initial data by two limits. The first one is the extreme limit

$$\mu \rightarrow 0. \quad (21)$$

In this limit the geometry of the Kerr black hole initial data changes from two asymptotically flat ends to one asymptotically flat and one cylindrical. The second limit is

$$s \rightarrow \infty. \quad (22)$$

This limit isolates the cylindrical structure of the extreme Kerr initial data cutting off the asymptotically flat end.

This procedure of taking the extreme limit can be performed for more generic data (see [25], [27]). And the behavior is identical, although, of course, one ends up with a different extreme throat initial data.

We isolate the properties of generic extreme throat initial data in the following definition. Consider the following Riemannian metric

$$h = \varphi^4 (e^{2q}(ds^2 + d\theta^2) + \sin^2 \theta d\phi^2), \quad (23)$$

where the functions φ and q depend only on θ . We assume that φ and q satisfy the following equation

$$\Delta_0 \varphi - \frac{1}{4}(1 - \partial_\theta^2 q)\varphi = -\frac{|\partial_\theta \omega|^2}{16 \sin^4 \theta \varphi^7}, \quad (24)$$

where Δ_0 is the Laplace operator in S^2 with respect to the standard metric acting on axially symmetric functions, namely

$$\Delta_0 \varphi = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \varphi). \quad (25)$$

Finally, for convenience, we define out of φ two additional functions, σ and η , as follows

$$\varphi^4 = e^\sigma, \quad \eta = \sin^2 \theta \varphi^4. \quad (26)$$

The function η is the square of the norm of the Killing vector (17) with respect to the metric (23).

With these ingredients, we can formulate the following definition.

Definition II.1. *Consider a set of functions (σ, ω, q) (depending only on θ) that satisfy equation (24) on S^2 and such that q vanished at the poles $\theta = 0, \pi$. Then, an extreme throat initial data set is a triple (S, h_{ij}, K_{ij}) where $S = \mathbb{R} \times S^2$, h_{ij} is given by (23) and K_{ij} is constructed from ω by the formula (16). In equation (16), the volume element ϵ_{ijk} is calculated with respect to the metric (23), the vector η^i is given by (17) and the indices are moved with the metric (23).*

From the definition it follows that the data satisfy the constraint equations (12)–(13), since equation (24) is just

the Lichnerowicz equation for the conformal factor φ with respect to the conformal metric

$$\tilde{h} = e^{2q}(ds^2 + d\theta^2) + \sin^2 \theta d\phi^2. \quad (27)$$

Note that the Ricci scalar of \tilde{h} is given by

$$\tilde{R} = 2e^{-2q}(1 - \partial_\theta^2 q). \quad (28)$$

For a discussion on the Lichnerowicz equation and the conformal method see, for example, the review article [6].

The vector η^i is a Killing vector of the metric h_{ij}

$$\mathcal{L}_\eta h_{ij} = 0, \quad (29)$$

where \mathcal{L} denotes Lie derivative. Moreover, the requirement that q vanishes at the poles arises from the regularity of the metric at the axis, namely the condition

$$\lim_{\theta \rightarrow 0, \pi} \frac{\partial_i \eta^i \partial^i \eta}{4\eta} = 1. \quad (30)$$

Hence, the metric h_{ij} is axially symmetric (see [22] [12] for a discussion of axial symmetry on initial data and also [36] for a general discussion of axial symmetry in General Relativity).

From the definition of K_{ij} it is also clear that η^i is a symmetry of K_{ij} , namely

$$\mathcal{L}_\eta K_{ij} = 0. \quad (31)$$

In our definition, we have made for simplicity the assumption that η^i is hypersurface orthogonal (with respect to the metric h_{ij}). We expect that all the results obtained in this article are also valid without this assumption, but this analysis remains to be done.

The metric h_{ij} has another symmetry, namely the vector ξ^i defined by

$$\xi^i = \frac{\partial}{\partial s}. \quad (32)$$

It straightforward to check that ξ^i is also a symmetry of K_{ij}

$$\mathcal{L}_\xi K_{ij} = 0. \quad (33)$$

Also, the vectors η^i and ξ^i commute.

Riemannian metrics of the form (23) are generically called cylindrically symmetric. In addition, we have equations (31) and (33) and hence at first sight it looks appropriate to call the whole initial data set cylindrically symmetric. However this terminology is misleading for the following reason: in general, the spacetime originated from this kind of data will not be cylindrically symmetric. Recall that a spacetime is cylindrically symmetric if it admits two spacelike commuting Killing vectors (see [36]). Since the problem of cylindrically symmetric spacetimes has been frequently analyzed in the literature, it is important to discuss this point in detail.

The vectors η^i and ξ^i are Killing vectors of h_{ij} and we also have equations (31)–(33), hence it follows from the results of [33] that the development of this class of initial data will be a spacetime with, at least, two Killing vectors. The projection of the spacetime Killing vectors to the initial surface are given by η^i and ξ^i (see [7] for a discussion about the relation of spacetime symmetries and symmetries on the initial data).

These data constitute initial data for the axially symmetric vacuum Einstein equations (see, for example, [20] [26]), hence it follows that the spacetime will be axially symmetric. In particular, the spacetime Killing vector η^μ corresponding to η^i will be spacelike outside the axis.

However, although the spacetime will have another symmetry it will not be cylindrically symmetric, because the extra symmetry will not be, in general, spacelike. The behavior of the spacetime Killing vector ξ^μ originated from the initial data symmetry ξ^i is clearly illustrated in the following explicit example which is also interesting by itself.

Consider the following 4-dimensional metric in coordinates (t, s, θ, ϕ)

$$g = \frac{(1 + \cos^2 \theta)}{2} \left[-\frac{e^{-2s}}{r_0^2} dt^2 + r_0^2 (ds^2 + d\theta^2) \right] + \eta_0 \left(d\phi + \frac{e^{-s}}{r_0^2} dt \right)^2, \quad (34)$$

where $r_0^2 = 2|J|$ and η_0 is given by (A9). This metric was introduced in [4] as the extreme Kerr throat geometry. It characterizes the spacetime geometry near the horizon of the extreme Kerr black hole.

It can be easily verified that the extreme Kerr throat initial data given by equations (20) and (16) are the initial data of the metric (34) in a surface $t = \text{constant}$. The spacetime Killing vectors of the metric g which correspond to the initial data Killing vectors η^i and ξ^i are given by

$$\eta^\mu = \frac{\partial}{\partial \phi}, \quad \xi^\mu = t \frac{\partial}{\partial t} - \frac{\partial}{\partial s}. \quad (35)$$

The metric has also two more Killing vectors (see [4])

$$\xi_1^\mu = \frac{\partial}{\partial t}, \quad \xi_2^\mu = \left(\frac{e^{-2s}}{2} + \frac{t^2}{2} \right) \frac{\partial}{\partial t} - t \frac{\partial}{\partial s} - e^{-s} \frac{\partial}{\partial \phi}. \quad (36)$$

We see that the Killing vector ξ^μ is not spacelike everywhere. In particular, the metric g is not cylindrically symmetric.

Finally, let us mention that there are two important physical quantities defined on a extreme throat initial data. First, the angular momentum given by

$$J = \frac{1}{8} (\omega(\pi) - \omega(0)). \quad (37)$$

This formula follows from the expression of the angular momentum for standard asymptotically flat axially symmetric initial data (see, for example, [22]). Second, the

area of the cylinder

$$A = 2\pi \int_0^\pi e^{\sigma+q} \sin \theta d\theta. \quad (38)$$

III. MAIN RESULT

The extreme limit procedure (21) and (22) that lead to the extreme throat initial data for the Kerr black hole discussed in the previous section II has an additional, remarkable property. The area of the extreme cylinder (with value $A = 8\pi|J|$) is smaller than the minimal surface area of any non-extreme Kerr black hole initial data (recall that the angular momentum J is kept fixed). In fact, the area of the minimal surface is a monotonically decreasing function with respect to μ . This can be, of course, trivially verified since for the Kerr black hole we have the explicit expression for A in terms of μ .

As we have pointed out, this extreme limit can be performed for other class of initial data, like the Bowen-York black hole initial data showed in section VI. It is conceivable (but it certainly remain to be shown) that there exists such procedure for general black hole initial data in axial symmetry, or at least for a relevant family of initial data. Let us assume that this is the case. That is, let us assume that for an initial data with an horizon of area A_1 we can perform the limit procedure to obtain an extreme throat initial data of area A , with $A \leq A_1$. Then, if inequality (10) is true, it should also holds for the extreme throat initial data. Our main result indicates that this is precisely the case. This result is summarized in the following theorem.

Theorem III.1. *Let us consider families of extreme throat initial data with fixed angular momentum J . Then, the area on these families satisfy the following properties:*

- *The first variation of the area is zero evaluated on the extreme Kerr throat initial data.*
- *The second variation of the area is positive evaluated on the extreme Kerr throat initial data.*

This theorem strongly suggests that the area is an absolute minimum for extreme Kerr throat initial data among all the extreme throat initial data with the same angular momentum. Since for extreme Kerr we have $A = 8\pi|J|$, the inequality (10) is satisfied for general extreme throat initial data. In order to prove that, we can follow a similar line as in [18] to prove that it is a local minimum and to [22] [15] [14] [13] to prove that it is in fact a global minimum. It appears that the same analysis will go through without major difficulties. This however should be checked and it will be done in a subsequent work.

Theorem III.1 gives also strong evidences in favor to inequality (10). Namely, if this inequality were false, there is no reason to expect that it will hold on extreme throat

initial data. As it have been pointed out above, this theorem suggest also an strategy to prove the conjecture: given an initial data with an apparent horizon construct a limit procedure analogous to (21) and (22) in such a way that i) in the limit an extreme throat initial data set is obtained and ii) the area of the extreme throat initial data is less or equal than the area of the horizon. In fact, in section VI we construct this limit procedure for the spinning Bowen-York family of initial data.

The proof of theorem III.1 is naturally divided in two parts, presented in sections IV and V respectively.

IV. THE MASS FUNCTIONAL FOR EXTREME THROAT INITIAL DATA

An extreme throat initial data are stationary if the following equations are satisfied

$$\Delta_0 \sigma - 2 = -\frac{|\partial_\theta \omega|^2}{\eta^2} \quad (39)$$

$$\Delta_0 \omega = 2 \frac{\partial_\theta \omega \partial_\theta \eta}{\eta}. \quad (40)$$

The fact that these equations for an extreme throat initial data define stationary solutions can be deduced from the standard stationary axially symmetric equations. However, for our present purpose, the only property of equations (39)–(40) that we will use is that the extreme Kerr throat initial data (defined by (A8)–(A7)) are a solution of them. This can be easily checked explicitly.

Equation (40) can be written in divergence form as follows

$$\partial_\theta \left(\sin \theta \frac{\partial_\theta \omega}{\eta^2} \right) = 0. \quad (41)$$

The stationary equations can be written in a natural form as equations on the unit sphere S^2 with the standard metric. Namely, let D_A be the covariant derivative with respect to the standard metric in S^2 . Then, equations (39)–(40) are written as

$$D_A D^A \sigma - 2 = \frac{D_A \omega D^A \omega}{\eta^2}, \quad (42)$$

$$D_A \left(\frac{D^A \omega}{\eta^2} \right) = 0. \quad (43)$$

These expressions were defined for axially symmetric functions, but they also make sense for functions which depends on ϕ . In fact, in all the results that follows we will not use the assumption that the functions are axially symmetric (this is very similar to what happens in the study of the inequality (2) discussed in [22]).

We define the following functional

$$\mathcal{M} = \int_0^\pi \left(|\partial_\theta \sigma|^2 + 4\sigma + \frac{|\partial_\theta \omega|^2}{\eta^2} \right) \sin \theta d\theta. \quad (44)$$

On the unit sphere, using the notation of equations (42)–(43) this functional is written as

$$\mathcal{M} = \frac{1}{2\pi} \int_{S^2} \left(|D\sigma|^2 + 4\sigma + \frac{|D\omega|^2}{\eta^2} \right) dS, \quad (45)$$

where $dS = \sin\theta d\theta d\phi$ is the volume element of the standard metric in S^2 . This functional is the obvious translation of the mass functional used in [19] adapted to this kind of initial data.

Let us make some general comments regarding the functional \mathcal{M} which are not directly relevant for the present article but they can have interesting future applications. It is very likely that for non-stationary initial data the functional \mathcal{M} represents a lower bound for another mass functional \mathcal{M}' which includes the time dependent terms. This is what happens with the functional considered in [19]. When the complete spacetime is considered (and not just the initial data), this new functional is precisely the total energy (the ADM mass) of axially symmetric, asymptotically flat spacetimes, and it is conserved (see [20]). In the present case, the mass functional \mathcal{M}' will describe the total energy of the class of spacetime discussed in section II. Namely, axially symmetric spacetimes which has another Killing vector. These spacetimes are not asymptotically flat. An analog situation occur for cylindrical symmetric spacetimes, for which the total energy can be defined (see [2] and reference therein). We emphasize however that the situation here is more complicated since the extra Killing vector is not spacelike everywhere. It would be very interesting to explore this issue and construct explicitly the functional \mathcal{M}' .

Relevant for our present purpose, are the following two important properties of the mass functional (44). We will prove them in lemma IV.1. The first one is that the stationary equations are the Euler-Lagrange equations of this functional. That is, the extreme Kerr throat initial data are critical points of this functional. The second property is that the second variation of this functional evaluated at the extreme Kerr throat initial data is positive. That suggests that extreme Kerr throat initial data set is in fact a minimum of this functional. These properties can be expected from the analysis developed in [19] and [18], since the functional \mathcal{M} is the natural generalization of the mass functional used in these articles adapted to cylindrical initial data.

Before proving that lemma is important to make the connexion between the functional \mathcal{M} and the energy of harmonic maps between S^2 and H^2 . Namely, consider the functional

$$\tilde{\mathcal{M}}_\Omega = \frac{1}{2\pi} \int_\Omega \frac{|\partial\eta|^2 + |\partial\omega|^2}{\eta^2} dS, \quad (46)$$

defined on some domain $\Omega \subset S^2$, such that Ω does not include the poles. Integrating by parts and using the identity

$$\Delta_0(\log(\sin\theta)) = -1, \quad (47)$$

we obtain the following relation between \mathcal{M} and \mathcal{M}'

$$\tilde{\mathcal{M}}_\Omega = \mathcal{M}_\Omega + 4 \int_\Omega \log \sin\theta dS + \oint_{\partial\Omega} (4\sigma + \log \sin\theta) \frac{\partial \log \sin\theta}{\partial n} ds, \quad (48)$$

where n denotes the exterior normal to Ω , ds is the surface element on the boundary $\partial\Omega$ and we have used the obvious notation \mathcal{M}_Ω to denote the mass functional (45) defined over the domain Ω . The difference between \mathcal{M} and \mathcal{M}' are the boundary integral plus the second term which is just a numerical constant. Note that if we integrate over S^2 this constant term is finite

$$\int_\Omega \log \sin\theta dS = 2 \log 2 - 2. \quad (49)$$

The boundary terms however diverges at the poles.

In an analogous way as it was described in [22], the functional \mathcal{M}' defines an energy for maps $(\eta, \omega) : S^2 \rightarrow \mathbb{H}^2$ where \mathbb{H}^2 denotes the hyperbolic plane $\{(\eta, \omega) : \eta > 0\}$, equipped with the negative constant curvature metric

$$ds^2 = \frac{d\eta^2 + d\omega^2}{\eta^2}. \quad (50)$$

The Euler-Lagrange equations for the energy \mathcal{M}' are called harmonic maps from $S^2 \rightarrow \mathbb{H}^2$. Since \mathcal{M} and \mathcal{M}' differ only by a constant and boundary terms, they have the same Euler-Lagrange equations.

We present in the following lemma the main result of this section.

Lemma IV.1. *Let us consider families of extreme throat initial data with fixed angular momentum J . Then, the area on these families satisfy the following properties:*

- *The first variation of \mathcal{M} is zero evaluated on the extreme Kerr throat initial data.*
- *The second variation of \mathcal{M} is positive evaluated on the extreme Kerr throat initial data.*

Proof. The proof follows very similar lines as the one presented in [18]. The only difference is the presence of an extra term in \mathcal{M} , the one containing σ . But this term, since it is linear, makes no contribution to the second variation which is the delicate part of the proof.

To define the variations, let us consider the real-valued function $\iota(\epsilon)$ defined by

$$\iota(\epsilon) = \mathcal{M}(\sigma(\epsilon), \omega(\epsilon)), \quad (51)$$

where

$$\sigma(\epsilon) = \sigma_0 + \epsilon \bar{\sigma}, \quad \omega(\epsilon) = \omega_0 + \epsilon \bar{\omega}. \quad (52)$$

We assume that $\bar{\omega}$ vanished at the poles $\theta = 0, \pi$. This boundary condition keeps fixed the angular momentum under the variations. In analogous way we define

$$\eta(\epsilon) = \sin^2 \theta e^{\sigma(\epsilon)}. \quad (53)$$

The first derivative of $\iota(\epsilon)$ with respect to ϵ is given by

$$\iota'(\epsilon) = \frac{1}{\pi} \int_{S^2} \{ D_A \sigma D^A \bar{\sigma} + 2\bar{\sigma} + (D_A \omega D^A \bar{\omega} - \bar{\sigma} |D\omega|^2) \eta^{-2} \} dS, \quad (54)$$

where a prime denotes derivative with respect to ϵ and the ϵ dependence in the right-hand side of (54) is encoded in the functions $\sigma(\epsilon), \omega(\epsilon), \eta(\epsilon)$ defined by (52)–(53). If we evaluate at $\epsilon = 0$, integrate by parts and use the condition that $\bar{\omega}$ vanished at the poles we obtain the Euler-Lagrange equations (42)–(43). Since extreme Kerr is a solution of this equation the first item in the Lemma is proved.

The second derivative of ι is given by

$$\iota''(\epsilon) = \frac{1}{16\pi} \int_{S^2} \{ |D\bar{\sigma}|^2 + (2\bar{\sigma}^2 |D\omega|^2 - 4\bar{\sigma} D_A \omega D^A \bar{\omega} + |D\bar{\omega}|^2) \eta^{-2} \} dS. \quad (55)$$

From equation (55), it is far from obvious that the second variation evaluated at the critical point $\epsilon = 0$ is positive definite. In order to prove that, the key ingredient is the following remarkable identity proved by Carter [10]. In terms of our variables it has the following form

$$F + \bar{\sigma} G'_\sigma + \bar{\omega} G'_\omega + 2\bar{\sigma} \bar{\omega} G_\omega - \eta^{-2} \bar{\omega}^2 G_\sigma = H \quad (56)$$

where

$$G_\sigma(\epsilon) = \Delta \sigma + \eta^{-2} |D\omega|^2 - 2, \quad (57)$$

$$G_\omega(\epsilon) = D_A (\eta^{-2} D^A \omega), \quad (58)$$

the derivatives with respect to ϵ of these functions are given

$$G'_\sigma(\epsilon) = \Delta \bar{\sigma} + (2D_A \bar{\omega} D^A \omega - 2\bar{\sigma} |D\omega|^2) \eta^{-2}, \quad (59)$$

$$G'_\omega(\epsilon) = D_A (\eta^{-2} (D^A \bar{\omega} - 2\bar{\sigma} D^A \omega)), \quad (60)$$

the positive definite function F is given by

$$F(\epsilon) = (D\bar{\sigma} + \bar{\omega} \eta^{-2} D\omega)^2 + (D(\bar{\omega} \eta^{-1}) - \eta^{-1} \bar{\sigma} D\omega)^2 + (\eta^{-1} \bar{\sigma} D\omega - \bar{\omega} \eta^{-2} D\eta)^2, \quad (61)$$

and the divergence term H is given by

$$H = D_A (\bar{\sigma} D^A \bar{\sigma} + \bar{\omega} \eta^{-1} D^A (\bar{\omega} \eta^{-1})), \quad (62)$$

The identity (56) is valid for arbitrary functions $\sigma, \omega, \bar{\sigma}, \bar{\omega}$ and it is straightforward to check although the computations are lengthy.

Note that using (59)–(60) and integrating by parts we obtain

$$- \int_{S^2} (\bar{\sigma} G'_\sigma(\epsilon) + \bar{\omega} G'_\omega(\epsilon)) dS = \pi \iota''(\epsilon). \quad (63)$$

We integrate on S^2 the identity (56). The divergence term H vanished (here we use again the boundary condition). We use (63) to obtain

$$\iota''(\epsilon) = \int_{S^2} F dS + \int_{S^2} (2\bar{\sigma} \bar{\omega} G_\omega(\epsilon) - \eta^{-2} \bar{\omega}^2 G_\sigma(\epsilon)) dS. \quad (64)$$

If we evaluate at $\epsilon = 0$ the last integral vanished, and hence we get the final result

$$\iota''(0) = \int_{S^2} F dS \geq 0. \quad (65)$$

□

The mass functional \mathcal{M} evaluated at extreme Kerr gives the value (A14) which in particular is not equal to the total mass m of extreme Kerr. This is to be expected since there is no obvious relation between \mathcal{M} and the total mass of the associated initial data with an asymptotically flat end and a cylindrical end. However, the value of \mathcal{M} at extreme Kerr suggests the following definition

$$m = C e^{\frac{\mathcal{M}}{16}}, \quad C = e^{-\frac{\ln(2)}{2} - \frac{1}{2}}. \quad (66)$$

We have normalized this quantity in such a way that gives the mass for extreme Kerr. It is also trivially positive definite (note that \mathcal{M} is not due to the extra term σ which has no sign). More important, the first variation of m and the second variation of m are given by

$$m' = 2^{-4} \mathcal{M}' m, \quad m'' = 2^{-8} (\mathcal{M}'' + (\mathcal{M}')^2) m. \quad (67)$$

And hence the functional m has the same critical points as \mathcal{M} and the second variation is also definitive positive. These properties makes the functional m attractive but we will not make use of it in the following. For the purpose of the proof of theorem III.1 only the functional \mathcal{M} is used.

V. VARIATION OF THE AREA FOR EXTREME THROAT INITIAL DATA

The results from previous section are somehow to be expected, since they are the analogous of the variational formulation presented in [19] and [18]. The remarkable new ingredient is the relation of this mass functional with the area. This is the subject of this section and it constitutes the most relevant part of this article.

Consider the formula for the area for an extreme throat initial data given by (38). The first and second variation of the area are given

$$A' = \int_{S^2} (\sigma' + q') e^{(\sigma+q)} dS, \quad (68)$$

and

$$A'' = \int_{S^2} ((\sigma' + q')^2 + (\sigma'' + q'')) e^{(\sigma+q)} dS. \quad (69)$$

In order to relate these equations with the mass functional we proceed as follows. We first write the Hamiltonian constraint (24) in terms of σ using the relation (26)

$$4\Delta_0\sigma + |\partial_\theta\sigma|^2 + \frac{|\partial_\theta\omega|^2}{\eta^2} - 4(1 - \partial_\theta^2 q) = 0. \quad (70)$$

We integrate this equation in S^2 . The first term gives zero. We write the second and third term in terms of mass functional (45), namely

$$\int_{S^2} |\partial_\theta\sigma|^2 + \frac{|\partial_\theta\omega|^2}{\eta^2} dS = 2\pi\mathcal{M} - 4 \int_{S^2} \sigma dS. \quad (71)$$

For the last term, we integrate by part the terms with $\partial_\theta^2 q$, namely

$$\int_{S^2} \partial_\theta^2 q dS = 2\pi \int_0^\pi \partial_\theta^2 q \sin \theta d\theta \quad (72)$$

$$= 2\pi \int_0^\pi (\partial_\theta(\partial_\theta q \sin \theta) - \partial_\theta q \cos \theta) d\theta \quad (73)$$

$$= -2\pi \int_0^\pi \partial_\theta q \cos \theta d\theta \quad (74)$$

$$= 2\pi \int_0^\pi (-\partial_\theta(q \cos \theta) + q \sin \theta) d\theta \quad (75)$$

$$= 2\pi \int_0^\pi q \sin \theta d\theta \quad (76)$$

$$= \int_{S^2} q dS. \quad (77)$$

To pass from (73) to (74) we have used that $\sin \theta$ vanished at $(0, \pi)$ and to pass from (75) to (76) we have used that q vanished at $(0, \pi)$. Collecting equations (71) and (77), from equation (70) we deduce our fundamental equation

$$\mathcal{M} = 8 + \frac{2}{\pi} \int_{S^2} (\sigma + q) dS. \quad (78)$$

From equation (78) we get an alternative expression for the first variation of the mass

$$\mathcal{M}' = \frac{2}{\pi} \int_{S^2} (\sigma' + q') dS. \quad (79)$$

And hence, using the first item in lemma IV.1, we get that

$$\int_{S^2} (\sigma' + q') dS|_{\epsilon=0} = 0. \quad (80)$$

Analogously, the second variation of the mass is given by

$$\mathcal{M}'' = \frac{2}{\pi} \int_{S^2} (\sigma'' + q'') dS. \quad (81)$$

Using the second item in lemma IV.1 we obtain

$$\int_{S^2} (\sigma'' + q'') dS|_{\epsilon=0} > 0. \quad (82)$$

We are now ready to compute the first and second variation of the area. If we evaluate the first variation of the area (equation (68)) at $\epsilon = 0$ and use the (remarkable) fact that $e^{\sigma_0 + q_0}$ is constant for the extreme Kerr cylinder (see equation (A10)) we get

$$A'|_{\epsilon=0} = 4|J| \int (\sigma' + q') dS|_{\epsilon=0}. \quad (83)$$

Using (80) we finally get

$$A'|_{\epsilon=0} = 0. \quad (84)$$

For the second variation we use equations (69) and again equation (A10) to obtain

$$A''|_{\epsilon=0} = 4|J| \int_{S^2} ((\sigma' + q')^2 + (\sigma'' + q'')) dS|_{\epsilon=0}. \quad (85)$$

The first term inside the integral is clearly positive definite and the second also by (81) and (82). Hence we deduce

$$A''|_{\epsilon=0} > 0. \quad (86)$$

This concludes the proof of theorem III.1.

VI. APPLICATION: SPINNING BOWEN-YORK INITIAL DATA

The Bowen-York initial data have been discovered in [9] and since that time they have been extensively used in both analytical and numerical studies. In this section we will prove that the area of the minimal surface (which is also an apparent horizon) for the family of spinning Bowen-York initial data satisfies the inequality (7). We will assume that the inequality is true for extreme throat initial data. As it was pointed out in section III theorem III.1 suggests that this is the case but the technical steps to complete the proof remain to be done.

The argument runs as follows. In [25] the extreme limit procedure was rigorously constructed for this kind of data. The only property of this limit not proved in this article was the monotonicity of the area. This is proved here as follows.

The area of any surface $r = \text{constant}$ is given by

$$A_\mu(r) = 2\pi r^2 \int_{S^2} \Phi_\mu^4 dS. \quad (87)$$

We use the same notation Φ_μ for the conformal factor of the Bowen-York family used in [25].

The location of the minimal surface (by the isometry of the data) is on $r = \mu/2$. That is, we want to consider the area $A_\mu(\mu/2)$. By definition of minimal surface, we known that

$$A_\mu(\mu/2) \leq A_\mu(r), \quad (88)$$

for all r . We also know that the conformal factor is monotonically decreasing with μ (Lemma 3.2 in [25]). That is, for $\mu_1 \leq \mu_2$ we have

$$\Phi_{\mu_1}(r, \theta) \leq \Phi_{\mu_2}(r, \theta). \quad (89)$$

Hence we have

$$A_{\mu_1}(r) \leq A_{\mu_2}(r). \quad (90)$$

Then we prove the following

$$A_{\mu_1}(\mu_1/2) \leq A_{\mu_1}(r) \leq A_{\mu_2}(r), \quad (91)$$

for all r . The first inequality in (91) follows from (88), and the second from (90). That is, we have proved that any surface for μ_2 has bigger area than the minimal surface for μ_1 . In particular, the minimal surface $r = \mu_2/2$, that is

$$A_{\mu_1}(\mu_1/2) \leq A_{\mu_2}(\mu_2/2). \quad (92)$$

Note, however, that inequality (91) is stronger than (92).

We have proved that the area of the minimal surface is monotonically decreasing under the extreme limit process constructed in [25]. And hence the area of the related extreme throat initial data (which can be also rigorously constructed, see [27] [28]) is smaller than the original area of the minimal surface. Since the inequality holds on the extreme cylinder it follows that it also holds for the spinning Bowen-York initial data.

VII. FINAL COMMENTS

The first open problem is to complete the analysis presented in the proof of theorem III.1 and prove the inequality (10) on extreme cylindrical initial data. We expect that the proof will follow similar lines as the ones presented in [18] [22] [15] [14] [13]. We are currently working on this [24].

The second open problem, which is much more difficult and relevant, is to construct an extreme limit procedure for generic axially symmetric initial data which satisfies the properties i) and ii) mentioned in section III. Then, the conjecture I.1 will be reduced to the extreme throat initial data case and hence it will be proved.

ACKNOWLEDGMENTS

It is a pleasure to thank Marc Mars for illuminating discussions regarding geometrical inequalities over many years. Particular useful for this work were the ones that took place at the Mathematisches Forschungsinstitut Oberwolfach during the workshop “Mathematical Aspects of General Relativity”, October 11th – October 17th, 2009 and during the conference “PDEs, relativity & nonlinear waves”, Granada, April 5-9, 2010. The author

thanks the organizers of these events for the invitation and the hospitality and support of the Mathematisches Forschungsinstitut Oberwolfach.

The author is supported by CONICET (Argentina). This work was supported in part by grant PIP 6354/05 of CONICET (Argentina), grant 05/B415 Secyt-UNC (Argentina) and the Partner Group grant of the Max Planck Institute for Gravitational Physics, Albert-Einstein-Institute (Germany).

Appendix A: The extreme Kerr cylindrical initial data

In this appendix we collect some well known properties of the extreme Kerr black hole initial data defined by a slice $t = \text{constant}$ in the Boyer-Lindquist coordinates. We use a slight modification of these coordinates. Let us denote by \tilde{r} the Boyer-Lindquist radius, we define $r = \tilde{r} - m$. In this way the cylindrical end is located at $r = 0$.

In the extreme case we have $\sqrt{|J|} = m$ (we always take the positive sign of the square root). We denote by a the standard angular momentum per unit mass parameter $a = J/m$. Note that for extreme Kerr we have two possible values for the angular momentum $J = \pm m^2$, and hence $a = \pm m$. The spacetime metric depends only one parameter, in our case it is appropriate to chose J as the free parameter.

The square of the norm η of the axial Killing vector is given by

$$\eta = \frac{((r + \sqrt{|J|})^2 + |J|)^2 - |J|r^2 \sin^2 \theta}{\Sigma} \sin^2 \theta, \quad (A1)$$

where Σ is given by

$$\Sigma = (r + \sqrt{|J|})^2 + |J| \cos^2 \theta. \quad (A2)$$

The twist potential of the axial Killing vector is given by

$$\omega = 2J(\cos^3 \theta - 3 \cos \theta) - \frac{2J|J| \cos \theta \sin^4 \theta}{\Sigma}. \quad (A3)$$

The conformal factor Φ and the function q that characterize the intrinsic metric (14) of the slice are given by

$$e^{2q} = \frac{\Sigma \sin^2 \theta}{\eta}, \quad \Phi^4 = \frac{\eta}{r^2 \sin^2 \theta}. \quad (A4)$$

From these function we compute the following limits

$$\lim_{r \rightarrow 0} (\sqrt{r} \Phi) = \varphi_0 = \left(\frac{4|J|}{1 + \cos^2 \theta} \right)^{1/4}, \quad (A5)$$

$$\lim_{r \rightarrow 0} e^q = e^{q_0} = \frac{1 + \cos^2 \theta}{2}, \quad (A6)$$

$$\lim_{r \rightarrow 0} \omega = \omega_0 = -\frac{8J \cos \theta}{1 + \cos^2 \theta}, \quad (A7)$$

The function σ_0 is given by

$$\sigma_0 = 4 \ln \varphi_0 = \ln(4|J|) - \ln(1 + \cos^2 \theta). \quad (A8)$$

We have the relation

$$\eta_0 = \sin^2 \theta e^\sigma = \sin^2 \theta \varphi_0^4. \quad (\text{A9})$$

From (A8) and (A6) we deduce the following key equation

$$e^{\sigma_0 + q_0} = 2|J|. \quad (\text{A10})$$

The mass functional \mathcal{M} defined by (44) evaluated at the extreme Kerr cylindrical initial data can also be calculated explicitly from these expressions. The calculus simplifies by noting that using equation (39) we can directly compute the integral

$$\int_0^\pi \left(\frac{|\partial_\theta \omega|^2}{\eta^2} \right) \sin \theta d\theta = 4. \quad (\text{A11})$$

The integral for σ_0 yields

$$\int_0^\pi \sigma_0 \sin \theta d\theta = 2 \ln(2) - 2 \ln(|J|) + 4 - \pi. \quad (\text{A12})$$

Finally, for the other integral we get

$$\int_0^\pi |\partial \sigma_0|^2 \sin \theta d\theta = -12 + 4\pi. \quad (\text{A13})$$

Hence, we obtain

$$\mathcal{M} = 8(\ln(2|J|) + 1). \quad (\text{A14})$$

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- [1] R. Arnowitt, S. Deser, and C. W. Misner. The dynamics of general relativity. In L. Witten, editor, *Gravitation: An Introduction to Current Research*, pages 227–265. Wiley, New York, 1962.
 - [2] Abhay Ashtekar, Jiri Bicak, and Bernd G. Schmidt. Asymptotic structure of symmetry reduced general relativity. *Phys. Rev.*, D55:669–686, 1997.
 - [3] Abhay Ashtekar Badri Krishnan. Isolated and dynamical horizons and their applications. *Living Reviews in Relativity*, 7(10), 2004.
 - [4] James M. Bardeen and Gary T. Horowitz. The extreme Kerr throat geometry: A vacuum analog of AdS(2) x S(2). *Phys. Rev.*, D60:104030, 1999.
 - [5] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure App. Math.*, 39(5):661–693, 1986.
 - [6] Robert Bartnik and Jim Isenberg. The constraint equations. In Piotr T. Chruściel and Helmut Friedrich, editors, *The Einstein equations and large scale behavior of gravitational fields*, pages 1–38. Birkhäuser Verlag, Basel Boston Berlin, 2004.
 - [7] Robert Beig and Piotr T. Chruściel. Killing initial data. *Classical Quantum Gravity*, 14(1A):A83–A92, 1997. Geometry and physics.
 - [8] Ivan Booth and Stephen Fairhurst. Extremality conditions for isolated and dynamical horizons. *Phys. Rev.*, D77:084005, 2008.
 - [9] Jeffrey M. Bowen and James W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21(8):2047–2055, 1980.
 - [10] B. Carter. Axisymmetric black hole has only two degrees of freedom. *Phys. Rev. Lett.*, 26(6):331–333, 1971.
 - [11] D. Christodoulou. Reversible and irreversible transformations in black-hole physics. *Phys. Rev. Lett.*, 25:1596–1597, 1970.
 - [12] Piotr T. Chruściel. Mass and angular-momentum inequalities for axis-symmetric initial data sets I. Positivity of mass. *Annals Phys.*, 323:2566–2590, 2008.
 - [13] Piotr T. Chruściel, Yanyan Li, and Gilbert Weinstein. Mass and angular-momentum inequalities for axis-symmetric initial data sets. II. Angular-momentum. *Ann. Phys.*, 323(10):2591–2613, 2008.
 - [14] Piotr T. Chruściel and Joao Lopes Costa. Mass, angular-momentum, and charge inequalities for axisymmetric initial data. *Class. Quant. Grav.*, 26:235013, 2009.
 - [15] Joao Lopes Costa. Proof of a Dain inequality with charge. *Journal of Physics A: Mathematical and Theoretical*, 43(28):285202, 2010.
 - [16] Sergio Dain. Initial data for a head on collision of two Kerr-like black holes with close limit. *Phys. Rev. D*, 64(15):124002, 2001.
 - [17] Sergio Dain. Angular momentum-mass inequality for axisymmetric black holes. *Phys. Rev. Lett.*, 96:101101, 2006.
 - [18] Sergio Dain. Proof of the (local) angular momentum-mass inequality for axisymmetric black holes. *Class. Quantum. Grav.*, 23:6845–6855, 2006.
 - [19] Sergio Dain. A variational principle for stationary, axisymmetric solutions of Einstein’s equations. *Class. Quantum. Grav.*, 23:6857–6871, 2006.
 - [20] Sergio Dain. Axisymmetric evolution of Einstein equations and mass conservation. *Class. Quantum. Grav.*, 25:145021, 2008.
 - [21] Sergio Dain. The inequality between mass and angular momentum for axially symmetric black holes. *International Journal of Modern Physics D*, 17(3-4):519–523, 2008.
 - [22] Sergio Dain. Proof of the angular momentum-mass inequality for axisymmetric black holes. *J. Differential Geometry*, 79(1):33–67, 2008.
 - [23] Sergio Dain and Maria E. Gabach Clement. Small deformations of extreme Kerr black hole initial data, 2010.
 - [24] Sergio Dain and Mara Eugenia Gabach Clément, 2010. In preparation.
 - [25] Sergio Dain and María Eugenia Gabach Clément. Extreme Bowen-York initial data. *Class. Quantum. Grav.*, 26:035020, 2009.
 - [26] Sergio Dain and Omar E. Ortiz. Well-posedness, linear perturbations, and mass conservation for the axisymmetric Einstein equations. *Phys. Rev. D*, 81(4):044040, Feb 2010.
 - [27] María Eugenia Gabach Clément. Conformally flat black hole initial data, with one cylindrical end, 2009.
 - [28] Mark Hannam, Sascha Husa, and Niall O Murchadha. Bowen-York trumpet data and black-hole simulations.

- Phys. Rev.*, D80:124007, 2009.
- [29] Jörg Hennig, Marcus Ansorg, and Carla Cederbaum. A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter. *Class. Quantum. Grav.*, 25(16):162002, 2008.
 - [30] Jörg Hennig, Carla Cederbaum, and Marcus Ansorg. A universal inequality for axisymmetric and stationary black holes with surrounding matter in the Einstein-Maxwell theory. *Commun. Math. Phys.*, 293:449–467, 2010.
 - [31] Pong Soo Jang. Note on cosmic censorship. *Phys. Rev. D*, 20(4):834–837, 1979.
 - [32] Marc Mars. Present status of the Penrose inequality. *Class. Quant. Grav.*, 26:193001, 2009.
 - [33] Vincent Moncrief. Spacetime symmetries and linearization stability of the Einstein equations. I. *J. Math. Phys.*, 16:493–498, 1975.
 - [34] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.
 - [35] Richard Schoen and Shing Tung Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, 79(2):231–260, 1981.
 - [36] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard Herlt. *Exact solutions of Einstein's field equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, second edition, 2003.
 - [37] László B. Szabados. Quasi-local energy-momentum and angular momentum in GR: A review article. *Living Rev. Relativity*, 7(4), 2004. cited on 8 August 2005.
 - [38] Gilbert Weinstein and Sumio Yamada. On a Penrose inequality with charge. *Commun. Math. Phys.*, 257(3):703–723, 2005.
 - [39] Edward Witten. A new proof of the positive energy theorem. *Commun. Math. Phys.*, 80(3):381–402, 1981.